Non-blocking networks based on expander graphs

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September 22, 2018
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Outline

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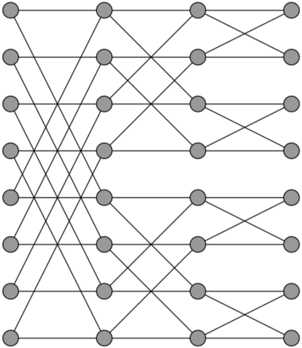
- Path selection algorithm

Some relevant networks: Butterfly



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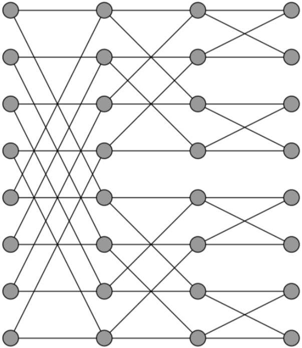

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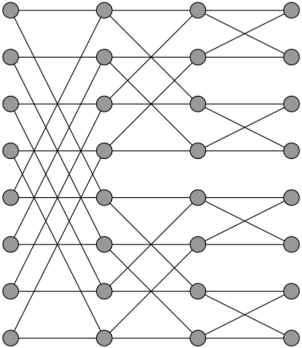

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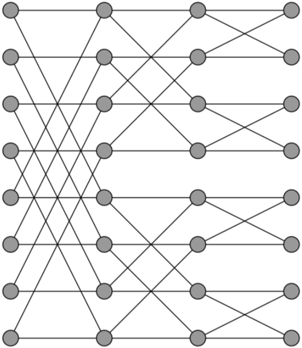

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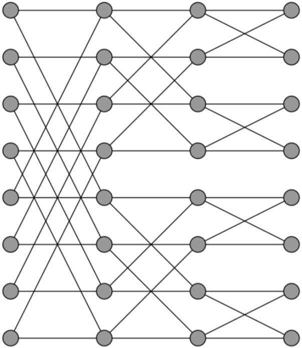

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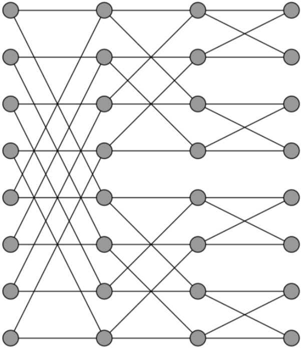

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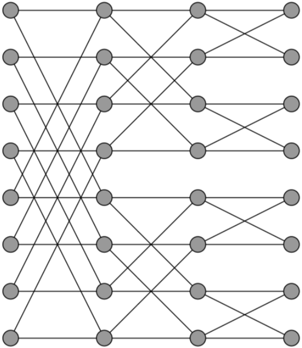

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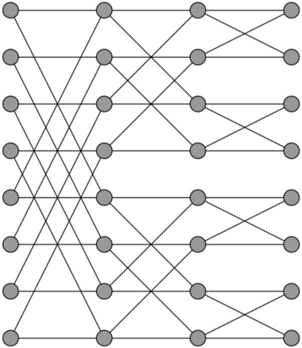

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1. If each i connects to random $\pi(i)$, then $N/\log N$ vertex disjoint paths can be established with high probability.
2. $\exists$ permutation π such that at most $\sqrt{N}$ vertex disjoint paths can be established.
3. Edge disjoint paths exist for $\pi(i) = i + \alpha \bmod N$, all α

Benes Network: $Be(n) = B(N) : B(N)^{-1}$

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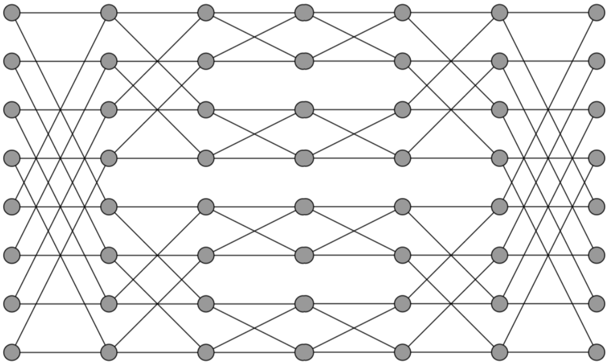

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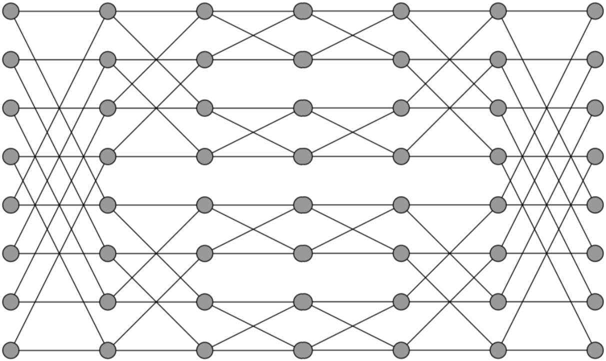
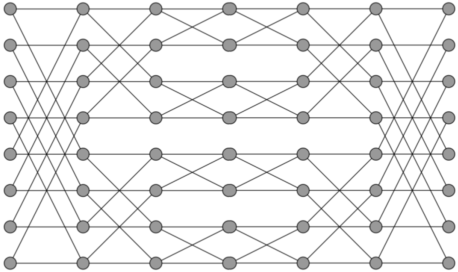


Theorem: We can establish vertex disjoint paths for any π.

Algorithm to make connections in Benes Network



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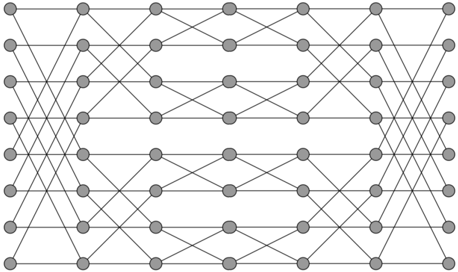

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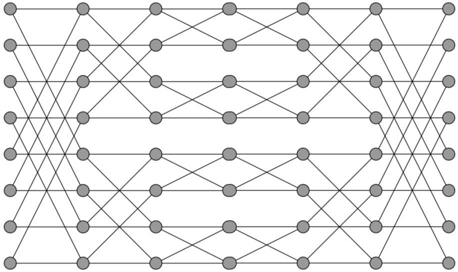

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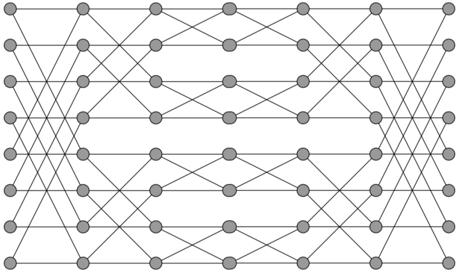

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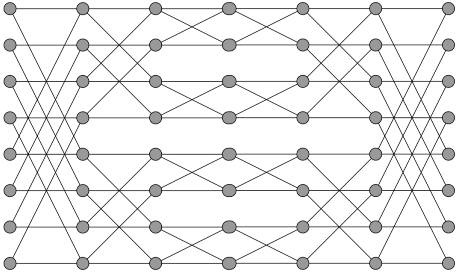

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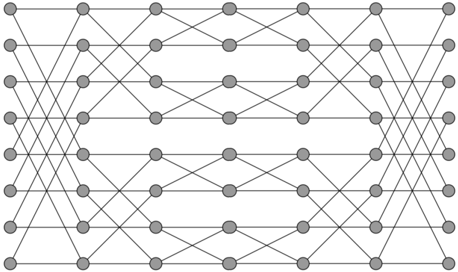

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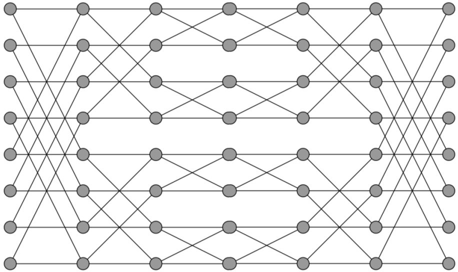

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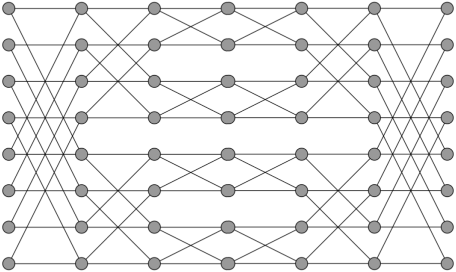
Similarly for the outputs.

$C(i) = i + \frac{N}{2},\ C(i + \frac{N}{2}) = i \quad i, C(i)$: competing pair.



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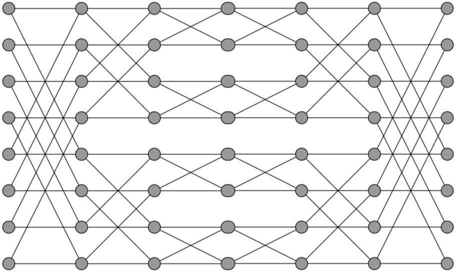

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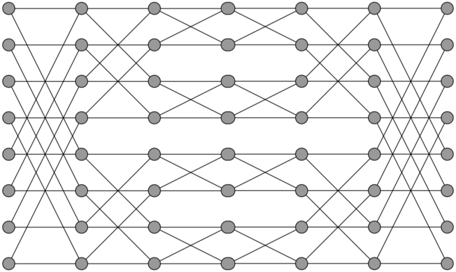

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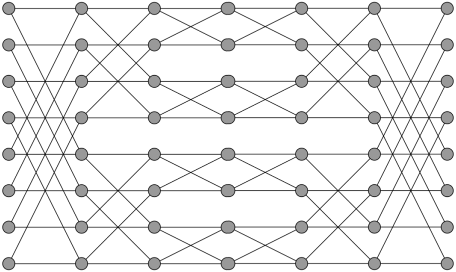

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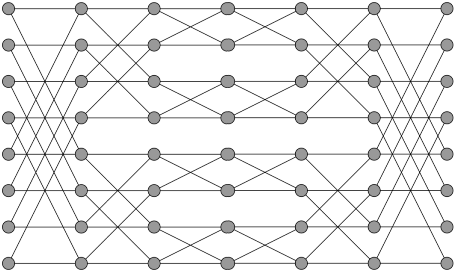

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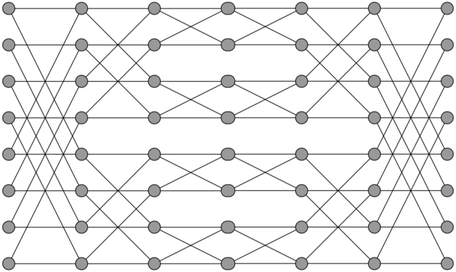

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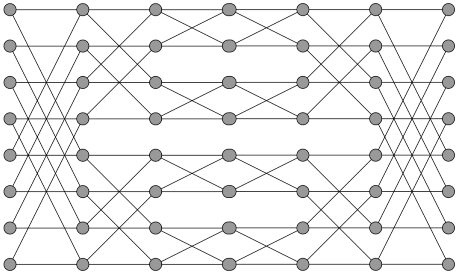

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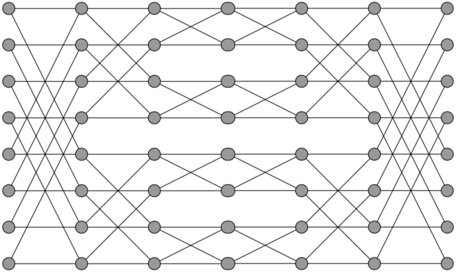

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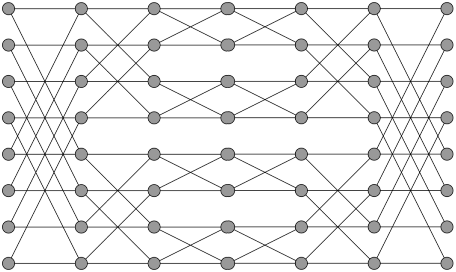

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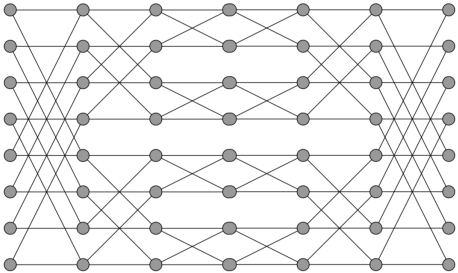

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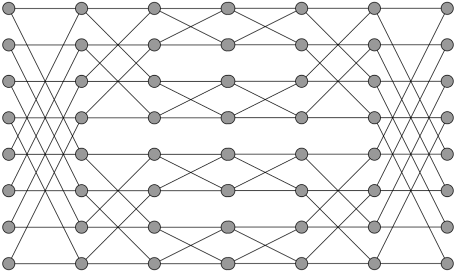

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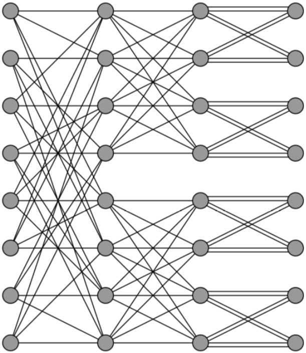
Multibutterfly $M_d(N)$



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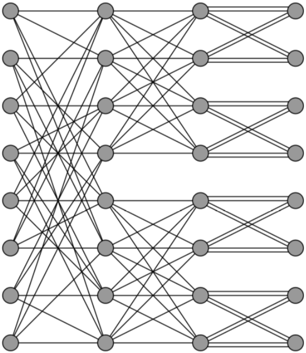

d edges to top and d edges to bottom from each node.

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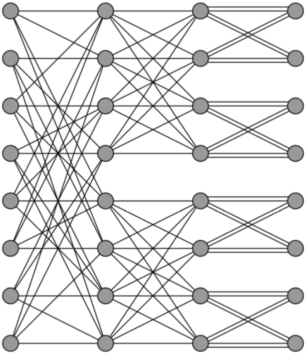

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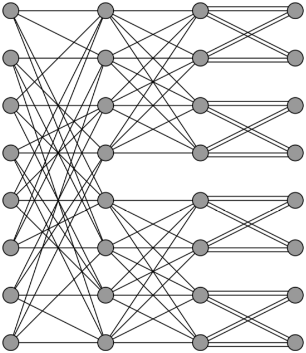

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Connections are random, s.t. all indegree, outdegree are $2d$.

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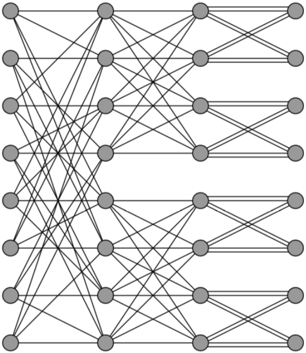

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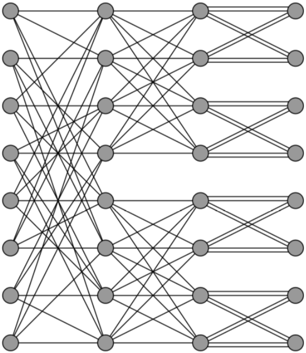

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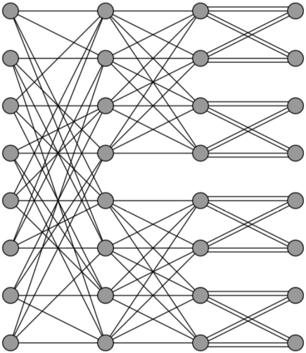

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Even for $\beta > 1$ $d = O(1)$ suffices.

(α, β, N, d) concentrator

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is a bipartite graph (U, V, E) where $|U| = N, |V| = N/2$ degree in U, V are resp. at most $d, 2d$, and every subset of $k \leq \alpha N$ nodes of U has edges to at least βk nodes of V.

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Theorem: Concentrator exists for $\alpha > 0$, $\beta > 1$ s.t. $\alpha\beta < 1$, and

$$d > \beta + 1 + \frac{\beta + 1 + \ln 4\beta}{ln(1/2\alpha\beta)}$$

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Proof: Randomized construction: Start with vertices U, V

(α, β, N, d) concentrator

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Concentrators connect level 0 to top and to bottor $M_d(N/2)$s

Theorem: Concentrator exists for $\alpha > 0$, $\beta > 1$ s.t. $\alpha\beta < 1$, and

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If you increase d, probability increases very rapidly.

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Simplifying gives the result.

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But will extending paths take time = O(finding matching)?

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- Requesters getting acknowledgement extend their path.
- Repeat.

Number of requests sent out from S is $d|S|$.
Number of requests received is at least $|T'| + 2(|T| - |T'|)$

$$d|S| \geq |T'| + 2(|T| - |T'|) = 2|T| - |T'|$$

But $|T| \geq \beta|S|$.

$|T'| \geq 2|T| - d|S| \geq 2\beta|S| - d|S| \geq |S|(2\beta - d)$.

The $|T'|$ acknowledgements must go to at least $|T'|/d$ requesters.

Number of successful requesters $\geq |S|(2\beta/d - 1)$.

Constant fraction progress if $2\beta > d$

Time per level $O(\log N)$, overall $O(\log^2 N)$.

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- Do not know what happens with higher radix Butterfly.

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